Simple proof of Bourgain's theorem on the singularity of the spectrum of Ornstein maps *

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Abstract. We give a simple proof of Bourgain's theorem on the singularity of Ornstein maps.

Setting and proof. Let (m_j), (t_j) be a sequence of positive integers, and let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space associated to Ornstein construction, that is,

$$\Omega = \prod_{j=1}^{+\infty} \big\{-t_j, \cdots, t_j\big\}^{p_j-1}, \quad \mathbb{P} = \bigotimes_{j=1}^{+\infty} \bigotimes_{k=1}^{p_j-1} \mathcal{U}_k,$$

where $\mathcal{U}_k$ is the uniform measure on $\big\{-t_j, \cdots, t_j\big\}^{p_j-1}$.

We want to prove that for almost all $\omega \in \Omega$, the spectral type μ_ω of the rank one map T_ω is singular.

Recall that μ_ω is the weak-star limit of the following sequence of probability measures

$$\Big(\prod_{j=1}^{N} |P_j(\omega, z)|^2 d\lambda\Big)_{N \geq 1},$$

where λ is the Lebesgue measure and for each $j \in \mathbb{N}^*$,

$$P_j(z) = \frac{1}{\sqrt{m_j}} \sum_{k=0}^{m_j-1} z^{n_{j,k}(\omega)},$$

$$n_{j,0}(\omega) = 0 \quad \text{and} \quad \text{for } \ k \geq 1, n_{j,k}(\omega) = k(h_j + t_j) + x_{j,k}(\omega) = k(h_j + t_j) + \omega_{j,k}.$$

We further assume that the sequence (m_j) is unbounded. Therefore, by Theorem 5.2 in "Calculus of Generalized Riesz Products"[1] combined with the uniform integrability of the sequence $\prod_{j=1}^{N} |P_j(\omega, z)|$, we have

$$\int_\Omega \int \prod_{j=1}^{N} |P_j(\omega, z)| dz d\mathbb{P} \xrightarrow[N \to +\infty]{} \int \sqrt{\frac{d\mu_\omega}{d\lambda}} d\mathbb{P},$$

We further have

$$\int |P_j(\omega, z)| d\mathbb{P} \xrightarrow[j \to +\infty]{} \frac{1}{2}\sqrt{\pi},$$

by the classical central limit theorem combined with the uniform integrability of the sequence $(|P_j(\omega, z)|)_{j \geq 0}$. We thus get

$$\int \prod_{j=1}^{N} |P_j(\omega, z)| d\mathbb{P} = \prod_{j=1}^{N} \int |P_j(\omega, z)| d\mathbb{P} \xrightarrow[N \to +\infty]{} 0.$$

Whence

$$\int \sqrt{\frac{d\mu_\omega}{d\lambda}} d\mathbb{P} = 0,$$

and the proof is complete.

*The reader need not be familiar with Ornstein construction neither the spectral theory of dynamical systems.

[1] Contemporary Mathematics(AMS) 631 (2014), pp. 145-180, by Nadkarni and the author.